UNITARY LIE GROUPS U(1) AND SU(2)

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Reference: Robert D. Klauber, *Student Friendly Quantum Field Theory, Vol. 2*, Section 2.2.7.

The Lie groups $U(1)$ and $SU(2)$ both play important roles in physics, so we'll examine the basics here. As a reminder, a Lie group is a group that depends on one or more continuous smoothly varying parameters. In physics, most groups are used in their matrix representations, that is, the group elements are represented by matrices and the group's binary operation is matrix multiplication. A unitary matrix is a matrix with complex elements whose hermitian transpose is also its inverse. That is, for a unitary matrix M, we have

$$M^{\dagger}=M^{-1} \tag{1}$$

so that

$$M^{\dagger}M=I \tag{2}$$

where I is the identity.

The notation $U(1)$ is the unitary group of degree 1, so its matrix representation consists of a single element. A suitable representation is

$$U(\theta)=e^{i\theta} \tag{3}$$

where θ is a real number. It is unitary, since $U^{\dagger}(\theta)=e^{-i\theta}=U^{-1}(\theta)$. It is a Lie group, since θ is a continuously varying quantity. For this group, the binary operation is ordinary multiplication. We can verify that this representation satisfies the group properties.

(1) The group is closed, since

$$e^{i\theta_1}\times e^{i\theta_2}=e^{i(\theta_1+\theta_2)} \tag{4}$$

and the RHS is also a member of the group.

(2) The identity element is with $\theta=0$, giving $U(0)=e^{i0}=1$.

(3) The inverse of $e^{i\theta}$ is $e^{-i\theta}$, since $e^{i\theta}e^{-i\theta}=e^{i0}=1$.

(4) The operation is associative, since multiplication of complex numbers is associative.

A unitary group becomes *special* if the determinants of all its matrix representations are $+1$. From this definition, we see that $U(1)$ is not special, since its determinant is just $e^{i\theta}$ itself, which is 1 only if $\theta = 0$.

An order 2 unitary group can be represented by a 2×2 matrix with complex elements. If we require this group to be special, this restricts the determinant to be $+1$. One such matrix representation is

$$M = \begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix} \tag{5}$$

where a and b are complex numbers. The special property is imposed by requiring that

$$aa^* + bb^* = 1 \tag{6}$$

We can see that M is unitary by calculating $M^\dagger M$. The hermitian conjugate $M^\dagger$ is complex conjugate or the transpose, so

$$M^\dagger = \begin{bmatrix} a^* & -b \\ b^* & a \end{bmatrix} \tag{7}$$

We then have

$$M^\dagger M = \begin{bmatrix} a^* & -b \\ b^* & a \end{bmatrix} \begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix} \tag{8}$$

$$= \begin{bmatrix} a^*a + bb^* & a^*b - ba^* \\ b^*a - ab^* & b^*b + aa^* \end{bmatrix} \tag{9}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{10}$$

where we used 6 on the diagonal elements.

To show that the set of matrices 5 form a group, we can verify the group properties.

(1) The group is closed as we can verify by matrix multiplication. We have

$$M_1 = \begin{bmatrix} a_1 & b_1 \\ -b_1^* & a_1^* \end{bmatrix} \tag{11}$$

$$M_2 = \begin{bmatrix} a_2 & b_2 \\ -b_2^* & a_2^* \end{bmatrix} \tag{12}$$

$$M_1 M_2 = \begin{bmatrix} a_1a_2 - b_1b_2^* & a_1b_2 + b_1a_2^* \\ -a_2b_1^* - a_1^*b_2^* & -b_1^*b_2 + a_1^*a_2^* \end{bmatrix} \tag{13}$$

$$\equiv \begin{bmatrix} a_3 & b_3 \\ -b_3^* & a_3^* \end{bmatrix} \tag{14}$$

where

$$\begin{aligned} a_3 &\equiv a_1 a_2 - b_1 b_2^* \\ b_3 &\equiv a_1 b_2 + b_1 a_2^* \end{aligned} \tag{15}$$

Thus $M_1 M_2$ has the same form as M in 5 so the binary operation is closed.

(2) The identity element has $a = 1$ and $b = 0$.
(3) The inverse element of M is $M^\dagger$ since the group is unitary.
(4) The group is associative, since matrix multiplication is associative.